THE CATEGORY OF SCHEMES IS NOT COCOMPLETE

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Abstract. Vladimir Voevodsky mentions that colimits do not exist in the category of schemes. This brief note gives a detailed proof of this fact.

1. Counter Example

Claim 1. *The category of schemes is not cocomplete.*

Proof. Let $R = k[x, y]/(xy - 1)$. Here we can take k to be any algebraically closed field not of characteristic 2. Observe that

$$k[x, y]/(xy - 1) \cong k[x][y]/(xy - 1) \cong k[x^{\pm 1}] \cong k[x]_{(x)}.$$

Since $k[x]$ is a principal ideal domain, its localization $k[x]_{(x)}$ is as well. The units $R^{\times}$ of the ring R are easily seen to be

$$\{rx^n | n \in \mathbb{Z}, r \in k^{\times}\}.$$

It follows that

$$\mathrm{Spec}(R) = \{(0), (x - a) | a \in k^{\times}\}.$$

Now let $L_1 = L_2 = \mathrm{Spec}(k[x])$. Notice

$$\mathrm{Spec}(k[x]) = \{(0), (x - a) | a \in k\}.$$

Consider the inclusion morphisms $R \xrightarrow{f_i} L_i$ given by

$$(x - a) \xmapsto{f_i} (x - ia)$$

$$(0) \xmapsto{f_i} (0).$$

We will prove the claim by contradiction. Suppose that there did indeed exist a coequalizer (P, f) to the diagram

$$R \rightrightarrows L_1 \coprod L_2.$$

By definition, $f \circ f_1(x) = f \circ f_2(x)$ for all $x \in \mathrm{Spec}(R)$. By our construction $f_1(x) = f_2(x) \iff x = (0)$. Moreover, $f_i(x) = (0) \iff x = (0)$. Therefore,

$$f_{|\mathrm{Spec}(R)}(x) = x \iff x = (0). \tag{1}$$

Notice that $L_1 \coprod L_2$ has two generic points $g_i = (0) \in L_i$. Let U_i be an affine open neighborhood around the image of g_i in P under the morphism f. Define $V_i = f^{-1}(U_i)$. V_i is an open affine in $L_1 \coprod L_2$ around the generic point g_i. Notice that

$$W = V_1 \cap V_2 = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2).$$

Since $W \cap L_i$ is a non-empty open in $\mathrm{Spec}(k[x])$, it follows that $W \cap L_i$ is dense in L_i, and hence $W \cap L_1 \cap L_2$ is dense in $\mathrm{Spec}(k[x])$. But this contradicts eq. (1). $\square$